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Energy transfer in the transient Ekman layer of a compressible fluid

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A theoretical analysis is made of the transient Ekman layer of a rapidly rotating compressible fluid. In the initial state, both the fluid and the disk are in equilibrium in isothermal rigid-body rotation. Flow is initiated by imposing a mechanical/thermal disturbance on the rotating disk. By making detailed examinations of the energy balance in the Ekman layer, the energy transfer mechanisms are delineated. Two distinctive transient energy transfer mechanisms are identified: (i) the one-dimensional energy diffusion process in the axial direction, and (ii) the conventional Ekman layer flow which is similar to that of an incompressible fluid. The usefulness of a particular grouping of flow variables, which is termed the energy flux content, is emphasized. The major distinctions between compressible and incompressible fluids are ascertained. This analytical endeavour clarifies the features unique to a compressible rotating flow. A unified view is constructed to encompass the previously published theoretical findings which had been presented in a piecemeal fashion.

1. Introduction

Transient flow of a compressible fluid over a rotating disk of infinite radius is considered. At the initial state, both the fluid and the disk move in unison and in rigid-body rotation about the central axis, which, for convenience, is aligned in the vertical direction (z^*). The fluid is in isothermal equilibrium with the disk at constant temperature T^*_{00}. To this initial state, a small thermal and/or mechanical disturbance is added to the rotating disk. The disturbance is represented as an arbitrary function of time and radial coordinate. The depiction of the subsequent fluid flow, in response to this externally prescribed disturbance, constitutes the theme of the present paper. It is stressed here that emphasis is placed on the transient features which are peculiar to a compressible fluid.

The present problem addresses a flow element basic to a more realistic situation involving a finite cylindrical container of height H^* and radius $r_0 H^*$. The bottom endwall disk rotates steadily with rotation rate Ω^*. The representative Ekman number $E[\equiv \mu/\rho^*_{00}(r_0 H^*)\Omega^* H^{*2}]$ is small, where $\rho^*_{00}(r_0 H^*)$ denotes the fluid density at the periphery and μ is the coefficient of shear viscosity. The present paper provides

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descriptions of energy transfers in the Ekman layer in the central region, for $E \ll 1$, apart from negligibly small zones near the axis and the cylindrical sidewall.

Knowledge of the compressible Ekman layer under discussion here is relevant to the analyses of gas centrifuges (Sakurai & Matsuda 1974). Under an extremely rapidly rotation of the system, i.e. $H^* \Omega^{*2}/g \gg 1$, where g denotes the gravitational acceleration, the conventional Boussinesq-fluid approximation is no longer applicable. In this situation, a formulation based on compressible-fluid flow is needed. In a pioneering paper, Sakurai & Matsuda (1974) analysed the thermally driven flow of a compressible fluid in a finite closed container. This study was subsequently extended and modified (e.g. see Conlisk 1985). A new approach was undertaken by Wood & Morton (1980) and by Barbarsky, Herbst & Wood (2002) based on Onsager's pancake model. They noted that, in rapidly rotating compressible fluids, much of the fluid mass is concentrated near the cylindrical sidewall. These accounts also emphasized the importance of compressible-fluid Ekman layer flows.

The prior studies on transient Ekman layers have been concentrated on the more familar case of an incompressible homogeneous (constant-density) fluid (e.g. Benton & Clark 1974; Zandbergen & Dijkstra 1987; Duck & Foster 2001). Also, much of the geophysical research employs an incompressible Boussinesq-fluid model in which density changes linearly with temperature (see Barcilon & Pedlosky 1967*a*, *b*; Sakurai 1969; Walin 1969; Homsy & Hudson 1971; Hyun, Fowlis & Warn-Varnas 1983). The character of the steady Ekman layer over timescales much larger than the rotational time (Ω^{*-1}) is qualitatively similar to both the constant-density model and the Boussinesq-fluid model. For a Boussinesq fluid, the horizontal force balance is between the Coriolis force and the viscous force in the Ekman layer, which is the case for a constant-density fluid as well. This is easily explained by noting that, in general, the lengthscale for the vertical temperature variation in the Boussinesq fluid is much larger than the Ekman layer thickness. Also, the transient Ekman layer of a Boussinesq fluid is formed at $t^* \sim O(\Omega^{*-1})$. In addition, the energy equation is reduced to a pure diffusion equation; therefore, the thermal diffusion effect is felt across the horizontal Ekman boundary layer (e.g. Holton 1965; Benton & Clark 1974; St-Maurice & Veronis 1975; Hyun 1984).

In contrast to these features of an incompressible flow, the distinctive aspects of a rapidly rotating compressible fluid flow will be explored. The principal difference stems from the fact that the compressible fluid motion in the radial direction gives rise to the generation (removal) of heat due to the compression (expansion) work. The velocity and temperature fields are strongly coupled in rapidly rotating compressible-fluid flows. This important flow character has been pointed out, in a piecemeal fashion, in a few studies (e.g. Riley 1967; Sakurai & Matsuda 1974; Matsuda & Hashimoto 1976; Harada 1979; Hyun & Park 1992; Lindbald, Bark & Zahrai 1994).

The present effort aims to provide a unified and systematic theoretical analysis of the transient Ekman layer of a compressible fluid. By developing a detailed energy balance formulation for a control volume, this study will explore the two evolutionary processes of energy transport mechanisms. One is akin to that of an incompressible fluid, in which the $O(1)$ radial motions and the concomitant $O(E^{1/2})$ Ekman-pumping take place. The other is unique to a compressible fluid, i.e. the energy transport by diffusion in the vertical direction across the Ekman layer. It will be shown that a combination of flow variables, $e[\equiv T + 2\alpha^2 rv]$, which will be termed the energy flux content, is effective in portraying the evolution of diffusive energy transport. The relevant governing equations will be scrutinized for the above two types of energy transfer mechanisms.

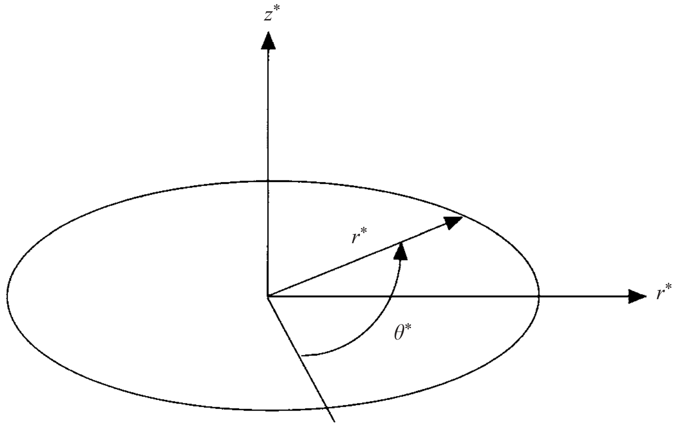

FIGURE 1. Coordinate system.

2. The mathematical model

The formulation is based on the cylindrical coordinate system (r^*, θ^*, z^*) with velocity components (u^*, v^*, w^*), rotating at rotation rate Ω^* (see figure 1). The conventional gravitational acceleration is dominated by the rotation effect and, therefore, is ignored. In the initial-state rigid-body rotation at temperature T^*_{00}, the radial density profile of the fluid is (e.g. Bark, Meijer & Cohen 1978; Park & Hyun 1998)

$$\rho^*_{00}(r^*) = \rho^*_{00}(r_0 H^*) \exp\left[\frac{\gamma M^2}{2}\left(r^2 - r_0^2\right)\right], \tag{1}$$

and the initial-state pressure distribution is

$$p^*_{00}(r^*) = \rho^*_{00}(r^*) R T^*_{00}. \tag{2}$$

In the above, subscript $*$ denotes dimensional quantities, subscript 00 refers to the basic state, $r \equiv r^*/H^*$, $M \equiv \Omega^* H^*/(\gamma R T^*_{00})^{1/2}$ is the system March number, γ the ratio of specific heats, and R the gas constant. It is implicit in the formulation that $M \sim O(1)$ such that the fluid compressibility is dominant.

Departures from the above initial-state isothermal rigid-body rotation are obtained by imposing small thermal and/or mechanical perturbations on the disk. The relative strength of perturbation is measured by the Rossby number $\varepsilon \equiv T^{*p}/T^*_{00}$ (or $U^{*p}/(\Omega^* H^*)$), where T^{*p} (or U^{*p}) indicates the magnitude of the thermal (or mechanical) perturbation at the disk. It follows that, for $\varepsilon \ll 1$, as viewed from the frame rotating at Ω^*, the dependent variables are $O(\varepsilon)$. Neglecting the $O(\varepsilon^2)$ and higher-order terms, the linearized governing equations in dimensional form are (e.g. Bark *et al.* 1978; Morberg, Hultgren & Bark 1984)

$$\frac{\partial \rho^*}{\partial t^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}(r^* \rho^*_{00} u^*) + \rho^*_{00}\frac{\partial w^*}{\partial z^*} = 0, \tag{3}$$

$$\rho^*_{00}\left(\frac{\partial u^*}{\partial t^*} - 2\Omega^* v^*\right) - \Omega^{*2} r^* \rho^* = -\frac{\partial p^*}{\partial r^*} + \mu^*\left[\left(\nabla^2 - \frac{1}{r^{*2}}\right)u^* + \left(\frac{1}{3} + \beta\right)\frac{\partial}{\partial r^*}(\nabla \cdot \boldsymbol{V}^*)\right], \tag{4}$$

$$\rho^*_{00}\left(\frac{\partial v^*}{\partial t^*}+2\Omega^* u^*\right)=\mu\left(\nabla^2-\frac{1}{r^{*2}}\right)v^*, \tag{5}$$

$$\rho^*_{00}\frac{\partial w^*}{\partial t^*}=-\frac{\partial p^*}{\partial z^*}+\mu\left[\nabla^2 w^*+\left(\frac{1}{3}+\beta\right)\frac{\partial}{\partial z^*}(\nabla\cdot\boldsymbol{V}^*)\right], \tag{6}$$

$$\rho^*_{00}C_p\frac{\partial T^*}{\partial t^*}-\frac{\partial p^*}{\partial t^*}-\Omega^{*2}r^*\rho^*_{00}u^*=k\nabla^2 T^*, \tag{7}$$

$$p^*=R(\rho^*_{00}T^*+\rho^* T^*_{00}). \tag{8}$$

In the above, C_p denotes the specific heat at constant pressure, β the ratio of expansion and shear viscosities, μ the coefficient of shear viscosity, k the coefficient of thermal conductivity.

The initial conditions are

$$\text{at } t^*=0, \quad u^*=v^*=w^*=T^*=0, \tag{9a}$$

and the boundary conditions at the disk are

$$\text{at } z^*=0, \quad u^*=w^*=0, \quad v^*=V^*_W(t^*,r^*), \quad T^*=T^*_W(t^*,r^*), \tag{9b}$$

in which $V^*_W(t^*,r^*)$ and $T^*_W(t^*,r^*)$ are the functional forms of the imposed perturbations.

As shown in (9*b*), in order to deal with general situations, the external perturbations V^*_W and T^*_W are assumed to be arbitrary functions of t^* and r^*. However, it is noted that certain restrictions are needed in this context. It is recalled that the formation time τ^* and thickness δ^*_H of the Ekman boundary layer are, respectively, $\tau^*\sim O(\Omega^{*-1})$ and $\delta^*_H\sim O(E^{1/2}H^*)$. It then follows that, in order for the linear Ekman layer equations to be applicable, the rates of temporal variation and of spatial variation of V^*_W and T^*_W should be smaller than $O(\Omega^*)$ and $O(\delta^{*-1}_H)$. The first implies that the period of temporal variation of the external perturbation at the disk should be larger than the Ekman layer formation time. Similarly, the second indicates that, for $E\ll 1$, the Ekman layer characteristics prevail, i.e. $\partial\Phi/\partial z^*\gg\partial\Phi/\partial r^*$, where Φ denotes a flow variable in the Ekman layer.

Finally, in the present study consideration is limited to axisymmetric flows, as expounded by Morberg *et al.* (1984) for the stability condition.

3. Energy transfer in the transient Ekman layer

Consider the horizontal layer adjacent to the disk, with the local coordinates and control volume as displayed in figure 2. The initial-state rigid-body rotation is described by (1) and (2). If a small perturbation ($\varepsilon\ll 1$) is applied to the rotating disk, the flow fields in the inertial coordinate frame are expressed, in dimensional form, as

$$\boldsymbol{V}^*_{inertial}=\Omega^* r^*\boldsymbol{e}_\theta+\varepsilon\left(u^{*p}_{inertial}\boldsymbol{e}_r+v^{*p}_{inertial}\boldsymbol{e}_\theta+w^{*p}_{inertial}\boldsymbol{e}_z\right), \tag{10a}$$

$$\rho^*=\rho^*_{00}+\varepsilon\rho^{*p}, \tag{10b}$$

$$T^*=T^*_{00}+\varepsilon T^{*p}, \tag{10c}$$

$$p^*=p^*_{00}+\varepsilon p^{*p}. \tag{10d}$$

In the above, $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_z)$ represent the unit vectors in the (r,θ,z) directions, superscripts $*$ and p denote, respectively, the dimensional and the perturbation quantities, and subscripts *inertial* and 00 stand for the inertial frame and the initial basic-state,

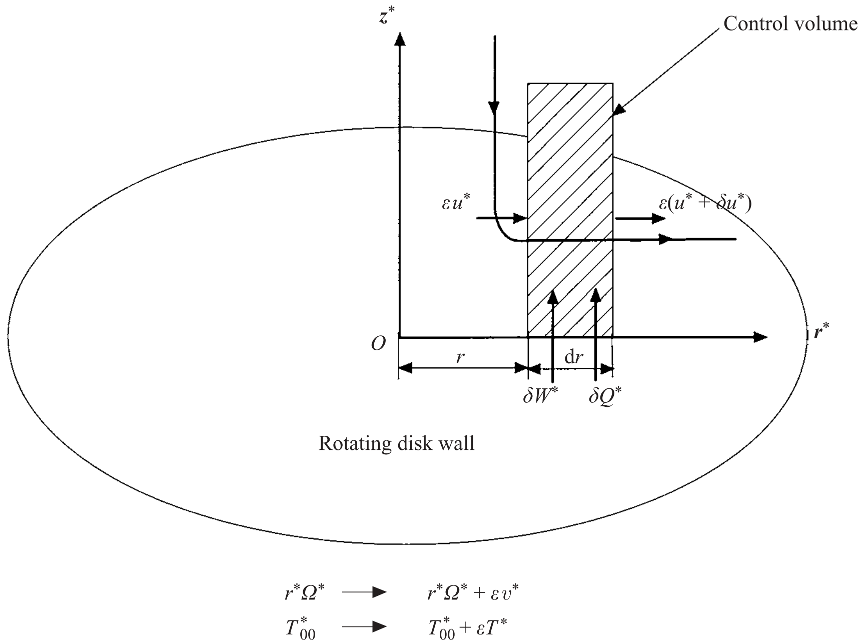

FIGURE 2. Schematic of the control volume in the Ekman layer.

respectively. It follows that, if (10*a*) is recast in the frame rotating with angular velocity Ω^*, the result is

$$\begin{aligned} V^*_{inertial} &= r^*\Omega^* \boldsymbol{e}_\theta + \boldsymbol{V}^*_{rotating}, \\ \text{where} \quad \boldsymbol{V}^*_{rotating} &= \varepsilon\left(u^{*p}_{rotating}\boldsymbol{e}_r + v^{*p}_{rotating}\boldsymbol{e}_\theta + w^{*p}_{rotating}\boldsymbol{e}_z\right), \end{aligned} \tag{11}$$

and subscript *rotating* denotes the rotating coordinates. From the above considerations, the perturbation velocities in the two frames are

$$u^{*p}_{rotating} = u^{*p}_{inertial}, \quad v^{*p}_{rotating} = v^{*p}_{inertial}, \quad w^{*p}_{rotating} = w^{*p}_{inertial}. \tag{12}$$

Now, a detailed examination is made of the balance of energy transport in the horizontal boundary layer on the disk. It is convenient to use the inertial frame. The small control volume consists of the annular zone in the horizontal boundary layer with inner and outer radii $(r^*, r^*+\mathrm{d}r^*)$, as shown in figure 2. The three components of transport of energy into and out of the control volume are: (i) the rate of mechanical work done by viscous friction at the control surface, δW^*_f; (ii) the rate of heat transfer due to the temperature gradient at the control surface, δQ^*; and (iii) the energy flux carried by the fluid motion across the control surface, δW^*_p. In the ensuing discussion, the thickness δ^*_H of the horizontal boundary layer is very small, i.e. $\delta^*_H \sim O(E^{1/2}H^*) \ll H^*$ (see e.g. Sakurai & Matsuda 1974; Harada 1979). Furthermore, it is noted that $\varepsilon \ll 1$ and the higher-order terms are ignored.

The mechanical work is done by the viscous frictional stress at the control surface. This stems from the viscous stress at the boundary of the control volume $[\boldsymbol{n}^* \cdot (\nabla \boldsymbol{V}^*_{inertial} + \{\nabla \boldsymbol{V}^*_{inertial}\}^T)]_{control\ surface}$, which is caused by the perturbation given to the disk. In the above, $\boldsymbol{n}^*$ is the outward unit vector normal to the control surface S^* as shown in figure 2, and T stands for transposition. Therefore, the rate of mechanical work, δW_f^*, transferred from the exterior to the interior of control volume is

$$\delta W_f^* = \mu \oint_{S^*} [\boldsymbol{n}^* \cdot (\nabla \boldsymbol{V}^*_{inertial} + \{\nabla \boldsymbol{V}^*_{inertial}\}^T)] \cdot \boldsymbol{V}^*_{inertial} \, \mathrm{d}S^*. \tag{13}$$

Using

$$\left| \frac{\partial v^{*p}_{rotating}}{\partial r^*} \Big/ \frac{\partial v^{*p}_{rotating}}{\partial z^*} \right| \sim O\left(E^{1/2}\right)$$

and (12), and substituting (10a) into (13), the integration can be approximated as

$$\delta W_f^* \cong \mu\varepsilon \left[\left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=z^*} - \left(\frac{\partial v^{*p}_{rotating}}{\partial z^*} \right)_{z^*=0} \right] (\Omega^* r^*) A^* + O\left(\varepsilon^2, \varepsilon E^{1/2}\right), \tag{14}$$

where z^* indicates an arbitrary axial position within the horizontal boundary layer, μ the coefficient of shear viscosity, and $A^* = 2\pi r^* \, \mathrm{d}r^*$ (see figure 2).

The heat transfer takes place due to the inequality in temperature across the control surface, i.e. the rate of heat transfer δQ^* transferred from the exterior to the interior of control volume, is

$$\delta Q^* = \varepsilon k \oint_{S^*} \boldsymbol{n}^* \cdot \nabla T^* \, \mathrm{d}S^*.$$

The rate of heat transfer through the horizontal surface of the control volume is an order-of-magnitude larger than through the vertical surface, i.e. $(\partial T^*/\partial r^*)/(\partial T^*/\partial z^*) \sim O(E^{1/2})$. Therefore, the above integration can be approximated as

$$\delta Q^* \cong \varepsilon k \left[\left(\frac{\partial T^{*p}}{\partial z^*} \right)_{z^*=z^*} - \left(\frac{\partial T^{*p}}{\partial z^*} \right)_{z^*=0} \right] A^* + O\left(\varepsilon E^{1/2}\right), \tag{15}$$

in which k is the coefficient of thermal conductivity.

The energy flux δW_p^* occurs by way of mass transport by the velocity normal to the control surfaces. The fluid energy per unit volume is written as

$$\begin{aligned} h^* &= \frac{\boldsymbol{V}^*_{inertial} \cdot \boldsymbol{V}^*_{inertial}}{2} + C_p T^* \\ &\cong \frac{(\Omega^* r^*)^2}{2} + C_p T^*_{00} + O(\varepsilon). \end{aligned}$$

Thus, the energy flux to the exterior from the interior of the control volume is

$$\delta W_p^* = \oint_{S^*} (\boldsymbol{V}^*_{inertial} \cdot \boldsymbol{n}^*) h^* \, \mathrm{d}S^*. \tag{16}$$

Upon placing (10a, c) and (12) into (16), the above integration for the control surface in figure 2 is approximately written as

$$\begin{aligned} \delta W_p^* = {} & \varepsilon \int_0^{z^*} \left[h^* \rho^*_{00} u^{*p}_{rotating} \right]_{r^*=r^*+\mathrm{d}r^*/2} 2\pi \left(r^* + \frac{\mathrm{d}r^*}{2} \right) \mathrm{d}z^* \\ & - \varepsilon \int_0^{z^*} \left[h^* \rho^*_{00} u^{*p}_{rotating} \right]_{r^*=r^*-\mathrm{d}r^*/2} 2\pi \left(r^* - \frac{\mathrm{d}r^*}{2} \right) \mathrm{d}z^* \\ & + \varepsilon \int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} \left[h^* \rho^*_{00} w^{*p}_{rotating} \right]_{z^*=z^*} 2\pi r^* \, \mathrm{d}r^* + O(\varepsilon^2), \end{aligned} \tag{17}$$

in which the first term on the right-hand side is the energy flux across the right vertical control surface, the second term is the energy flux across the left vertical control surface, and the third term the energy flux through the top horizontal control surface. For the integration of the third term on the right-hand side of (17), consider the continuity equation:

$$\left[r^* \rho_{00}^* w_{rotating}^{*p}\right]_{z^*=z^*} = -\int_0^{z^*} r^* \frac{\partial \rho^{*p}}{\partial t^*} \, \mathrm{d}z^* - \int_0^{z^*} \frac{\partial \left(r^* \rho_{00}^* u_{rotating}^{*p}\right)}{\partial r^*} \, \mathrm{d}z^*. \qquad (18)$$

Substituting (18) into the third term on the right-hand side of (17), and integrating by parts gives

$$\begin{aligned} &\varepsilon \int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} \left[h^* \rho_{00}^* w_{rotating}^{*p}\right]_{z^*=z^*} 2\pi r^* \, \mathrm{d}r^* \\ &= -\varepsilon \int_0^{z^*} \left[h^* \rho_{00}^* u_{rotating}^{*p} 2\pi r^*\right]_{r^*=r^*+\mathrm{d}r^*/2} \mathrm{d}z^* \\ &\quad + \varepsilon \int_0^{z^*} \left[h^* \rho_{00}^* u_{rotating}^{*p} 2\pi r^*\right]_{r^*=r^*-\mathrm{d}r^*/2} \mathrm{d}z^* \\ &\quad + \varepsilon \int_0^{z^*} \int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} \Omega^{*2} r^{*2} \rho_{00}^* u_{rotating}^{*p} 2\pi \, \mathrm{d}r^* \, \mathrm{d}z^* \\ &\quad - \varepsilon \int_0^{z^*} \int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} h^* r^* \frac{\partial \rho^{*p}}{\partial t^*} 2\pi \, \mathrm{d}r^* \, \mathrm{d}z^* + O(\varepsilon^2). \end{aligned} \qquad (19)$$

From (17) and (19), the rate of energy flux δW_p^* becomes

$$\begin{aligned} \delta W_p^* \cong{}& \varepsilon \int_0^{z^*} \Omega^{*2} r^* \rho_{00}^* u_{rotating}^{*p} \, \mathrm{d}z^* A^* \\ &- \varepsilon \int_0^{z^*} \int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} h^* r^* \frac{\partial \rho^{*p}}{\partial t^*} 2\pi \, \mathrm{d}r^* \mathrm{d}z^* + O(\varepsilon^2, \varepsilon(\mathrm{d}r^*)^*). \end{aligned} \qquad (20)$$

For further manipulation of (20), the θ-momentum equation in the rotating frame (see (5)) is rewritten, in dimensional form, as

$$2\rho_{00}^* \Omega^* u_{rotating}^{*p} = -\rho_{00}^* \frac{\partial v_{rotating}^{*p}}{\partial t^*} + \mu \left(\frac{\partial^2 v_{rotating}^{*p}}{\partial r^{*2}} + \frac{\partial^2 v_{rotating}^{*p}}{\partial z^{*2}} - \frac{v_{rotating}^{*p}}{r^{*2}} \right). \qquad (21)$$

Noting that the thickness of the horizontal boundary layer is $\delta_H^* \sim O(E^{1/2} H^*)$, estimations can be made of the magnitudes of the terms of (21):

$$\left| \frac{\partial^2 v_{rotating}^{*p} / \partial r^{*2}}{\partial^2 v_{rotating}^{*p} / \partial z^{*2}} \right| \sim O(E),$$

$$\left| \frac{v_{rotating}^{*p} / r^{*2}}{\partial^2 v_{rotating}^{*p} / \partial z^{*2}} \right| \sim O(E).$$

As a result, (21) is reduced to

$$2\rho_{00}^* \Omega^* u_{rotating}^{*p} \cong -\rho_{00}^* \frac{\partial v_{rotating}^{*p}}{\partial t^*} + \mu \frac{\partial^2 v_{rotating}^{*p}}{\partial z^{*2}} + O(E). \qquad (22)$$

Substituting (22) into (20) yields

$$\delta W_p^* \cong \varepsilon \frac{\mu}{2}\left[\left(\frac{\partial v_{rotating}^{*p}}{\partial z^*}\right)_{z^*=z^*} - \left(\frac{\partial v_{rotating}^{*p}}{\partial z^*}\right)_{z^*=0}\right](\Omega^* r^*)A^* - \varepsilon\frac{1}{2}\int_0^{z^*} \rho_{00}^* \frac{\partial v_{rotating}^{*p}}{\partial t^*}\,\mathrm{d}z^*(\Omega^* r^*)A^* - \varepsilon\int_0^{z^*} h^*\frac{\partial \rho^{*p}}{\partial t^*}\,\mathrm{d}z^* A^* + O(\varepsilon^2, \varepsilon E, \varepsilon(\mathrm{d}r^*)^2). \quad (23)$$

It is worth noting that the sum of the first and second terms on the right-hand side of (23) for δW_p^* is the same as the rate of pressure work. In the present analysis, since the control volume is fixed in space, no explicit mention of the rate of pressure work is made. In the Lagrangian formulation considering a material volume, the term representing the rate of pressure work appears explicitly. The work is done by the pressure gradient when the fluid particle undergoes radial motions in the pressure field which has been set by the initial-state pressure distribution of (2), i.e.

$$\text{rate of pressure work} \cong \varepsilon \int_0^{\delta_H^*} u_{rotating}^{*p} \frac{\mathrm{d}p_{00}^*}{\mathrm{d}r}\,\mathrm{d}z^* A^*.$$

It is a straightforward task to verify that substituting (2) and (22) into the above equation leads to the same result.

The rate of change of the total energy in the control volume, $\partial E^*/\partial t^*$, is

$$\frac{\partial E^*}{\partial t^*} = \int_0^{z^*}\int_{r^*-\mathrm{d}r^*/2}^{r^*+\mathrm{d}r^*/2} \frac{\partial(h^*\rho^*)}{\partial t^*} 2\pi r^*\,\mathrm{d}r^*\,\mathrm{d}z^* \cong \varepsilon\int_0^{z^*} \rho_{00}^*\left(\Omega^* r^* \frac{\partial v_{rotating}^{*p}}{\partial t^*} + C_p^*\frac{\partial T^{*p}}{\partial t^*}\right)\mathrm{d}z^* A^* + \varepsilon\int_0^{z^*} h^*\frac{\partial\rho^{*p}}{\partial t^*}\,\mathrm{d}z^* A^* + O(\varepsilon^2, \varepsilon(\mathrm{d}r^*)^2). \quad (24)$$

In order to fulfil the energy conservation in the above control volume, the following relation should be satisfied:

$$\frac{\partial E^*}{\partial t^*} = \delta W_f^* + \delta Q^* - \delta W_p^*. \quad (25)$$

Under the assumptions that $\varepsilon \ll 1$, $E \ll 1$, $\mathrm{d}r^* \ll 1$, substituting (14), (15), (23) and (24) into (25) produces

$$\int_0^{z^*} \rho_{00}^*\left(\frac{\Omega^* r^*}{2}\frac{\partial v_{rotating}^{*p}}{\partial t^*} + C_p^*\frac{\partial T^{*p}}{\partial t^*}\right)\mathrm{d}z^* = \int_0^{z^*}\frac{\partial^2 e^*}{\partial z^{*2}}\,\mathrm{d}z^*, \quad (26)$$

in which

$$e^* = \mu\frac{\Omega^* r^*}{2} v_{rotating}^{*p} + kT^{*p}. \quad (27)$$

By combining the foregoing mathematical developments, it is obvious that, in the time-dependent Ekman boundary layer,

$$\rho_{00}^*\left(\frac{\Omega^* r^*}{2}\frac{\partial v_{rotating}^{*p}}{\partial t^*} + C_p\frac{\partial T^{*p}}{\partial t^*}\right) = \frac{\partial^2 e^*}{\partial z^{*2}}. \quad (28)$$

From (28), in the steady state, e^* is independent of z^*, i.e. the value of e^* in the Ekman boundary layer is uniform in z^*, maintaining the value of e^* at the disk. It then follows that

$$e^*\left[\equiv\tfrac{1}{2}\Omega^* r^* \mu v^*_{rotating} + kT^*\right] = \tfrac{1}{2}\mu\Omega^* r^* V^*_W + kT^*_W. \tag{29}$$

In the above, as indicated previously, V^*_W and T^*_W respectively stand for the perturbed azimuthal velocity and the perturbed temperature at the disk viewed in the rotating frame. Clearly, in the steady Ekman layer case, V^*_W and T^*_W are not functions of time. The newly found combination of flow variables, e^* in (27), will for convenience be termed the energy flux content. The significance of this variable grouping will be obvious in theoretical analyses of the fundamental flow properties of a rapidly rotating compressible fluid. The energy flux content e^* turns out to be a powerful and effective tool in the analysis of energy transport in the Ekman layer as well as in general dynamical considerations for a rotating compressible fluid.

A rudimentary form of the energy flux content (e^*) was pointed out earlier in Sakurai & Matsuda (1974) and Wood & Morton (1980), although not in a direct and explicit manner.

In the next section of this paper, the concrete physical meaning and the usefulness of e^* will be elaborated. This shows a close correlation between the energy flux content and the variables in previous papers.

4. The physical implications of e^*

The physical meaning of the energy flux content e^* will now be clarified. For this purpose, a non-dimensional formulation will be introduced. A similar non-dimensional variable grouping, $T + 2\alpha^2 rv$, in which $\alpha^2 \equiv \sigma(\gamma-1)M^2/4r_0^2$, emerged in the steady-state analysis of gas centrifuge flows (see (3.20) of Sakurai & Matsuda 1974). Also, in the early analysis using the Onsager pancake model, another similar variable grouping, $h(\equiv \theta + (S-1)\omega)$ (see p. 5 in Wood & Morton 1980), was obtained. This variable grouping was derived in the course of extensive mathematical developments of the above theoretical expositions. In the present paper, an effort is made to delineate the signficance and physical interpretation of this variable grouping, the energy flux content e^*.

As asserted previously, in the steady state, the energy flux content e^* is uniform in the z^*-direction over the entire thickness of the horizontal boundary layer. As shown in (28), the perturbation imposed at the disk propagates in the z^*-direction in a one-dimensional diffusion-type process. To gain a clearer picture, non-dimensionalization is implemented (e.g. Hultgren, Meijer & Bark 1981; Park & Hyun 1998):

$$\begin{aligned} r &= r^*/H^*,\\ t &= t^*\Omega^*,\\ v &= v^{*p}_{rotating}\big/\Omega^* H^*,\\ T &= T^{*p}/T^*_{00},\\ \rho_{00} &= \rho^{*p}_{00}\big/\rho^*_{00}(r_0 H^*). \end{aligned}$$

Then, (28) reduces to

$$\rho_{00}\frac{\partial}{\partial t}(\sigma T + 2\alpha^2 rv) = E\frac{\partial^2 e}{\partial z^2}. \tag{30}$$

In the above, $\sigma \equiv \mu C_p/k$ is the Prandtl number, and $e[\equiv e^*/(kT^*_{00})] \equiv T + 2\alpha^2 rv$ is the non-dimensional energy flux content. It follows that, if $\sigma = 1.0$, the energy flux content e satisfies the one-dimensional diffusion equation, i.e.

$$\rho_{00}\frac{\partial e}{\partial t}=\frac{\partial^2 e}{\partial \eta^2}, \tag{31}$$

in which η denotes the boundary-layer coordinate, $\eta \equiv z/E^{1/2}$. Equation (31) demonstrates that, when a perturbation is given to the disk, the disturbance propagates in the η-direction in a one-dimensional diffusion process. In the steady state, after a sufficient time has elapsed, therefore, the energy flux content e becomes uniform in the η-direction over the thickness of the horizontal boundary layer. This finding is consistent with the prior observation of Sakurai & Matsuda (1974), which showed that, in the steady horizontal boundary layer,

$$\tilde{T}+2\alpha^2 r\tilde{v}=0, \tag{32}$$

in which the tilde represents the boundary-layer variable in the boundary-layer matching method.

It is of interest to note that the above-described energy flux content e is uniform across the steady Stewartson layer of a compressible rotating flow. (The Stewartson layer forms on the cylinder sidewall which is parallel to the rotation axis.) This has been pointed out in a number of different contexts (e.g. Bark & Bark 1976; Matsuda & Hashimoto 1976; Park & Hyun 1997, 1998). It may suggest the possibility of an energy transfer mechanism in the transient Stewartson layer. However, detailed discussions on the Stewartson layer are beyond the scope of the present paper, and they will not be pursued here.

A note is in order on the case when the disk wall is adiabatic, i.e. $(\partial\tilde{T}/\partial\eta)_{\eta=0}=0$. It then follows from the formulation of (32) that $(\partial\tilde{e}/\partial\eta)_{\eta=0}=0$, and, subsequently, $(\partial\tilde{v}/\partial\eta)_{\eta=0}=0$. This implies that, for $(\partial\tilde{v}/\partial\eta)_{\eta=0}=0$, no Ekman layer flow of $\tilde{v}\sim O(1)$ at $\eta\sim O(1)$ can exist. This leads to the conclusion that, for an adiabatic disk wall, a much weaker flow of $\tilde{v}\sim O(E^{1/2})$ at $\eta\sim O(1)$ exists. This finding is in line with the results of preceding studies (e.g. Matsuda & Hashimoto 1976; Bark & Hultgren 1979; Lindbald *et al.* 1994). Furthermore, the present discussion reinforces the usefulness of the energy flux content e in deducing physical rationalizations of the flow behaviour.

The physical meaning of the energy flux content e is now illuminated in further detail. For a compressible fluid, if a thermal (or mechanical) perturbation is imposed on the disk, energy transport from the disk to the fluid in the z-direction, as is evident in (30) and (31), takes place. It is important to note that the energy thus transported to the fluid represents only a portion of the energy which was originally supplied to the disk from the exterior. This argument is obvious by working with (28) and (29). Suppose that, for the fluid at the initial-state isothermal rigid-body rotation ($T^*=T^*_{00}$, $v^*_{inertial}=r^*\Omega^*$), a mechanical perturbation (εV^*_W) and/or a thermal perturbation (εT^*_W) is imposed on the horizontal disk. In response to this, the perturbed mechanical energy (PME) and the perturbed thermal energy (PTE) per unit volume of the fluid in the boundary layer are, respectively,

$$\begin{aligned}\mathrm{PME}&=\tfrac{1}{2}(\rho^*_{00}+\varepsilon\rho^{*p})(r^*\Omega^*+\varepsilon v^*_{rotating})^2-\tfrac{1}{2}\rho^*_{00}(r^*\Omega^*)^2\\ &\simeq \underbrace{\varepsilon\rho^*_{00}(r^*\Omega^*)v^{*p}_{rotating}}_{\mathrm{I}}+\underbrace{\varepsilon\tfrac{1}{2}\rho^{*p}(r^*\Omega^*)^2}_{\mathrm{II}}+O(\varepsilon^2),\end{aligned} \tag{33}$$

$$\begin{aligned}\mathrm{PTE}&=C_p(\rho^*_{00}+\varepsilon\rho^{*p})(T^*_{00}+\varepsilon T^{*p})-C_p\rho^*_{00}T^*_{00}\\ &\simeq \underbrace{\varepsilon C_p\rho^*_{00}T^{*p}}_{\mathrm{III}}+\underbrace{\varepsilon C_p\rho^{*p}T^*_{00}}_{\mathrm{IV}}+O(\varepsilon^2).\end{aligned} \tag{34}$$

In view of (28), (33) and (34), the left-hand side of (28) denotes the temporal rate of change of ($\frac{1}{2} \times \mathrm{I} + \mathrm{II}$), which is a portion of the total perturbed energy (= PME + PTE). The right-hand side of (28) indicates the diffusion rate of the energy flux content e^*. For $\sigma = 1.0$, $\frac{1}{2} \times \mathrm{I} + \mathrm{III} = e^*$; this implies that (31) is the equation describing the temporal rate of change of e^*. Note, again, that e^* represents only a portion of the total perturbed energy. In summary, in the Ekman layer flow of a rapidly rotating compressible fluid, only a portion of the energy that was supplied at the disk is transferred to the fluid in the z^*-direction in a one-dimensional diffusion process. It is asserted here that the amount of the energy thus transported is effectively described by the physical variable, e^*.

On the other hand, the remainder of the energy, i.e. ($\frac{1}{2} \times \mathrm{I} + \mathrm{II} + \mathrm{IV}$), is used, in the form of horizontal energy flux, to give rise to the horizontal flows. This aspect will be further clarified in the ensuing discussion. Now, specific evaluations will be made of the magnitudes of the energy diffused in the z^*-direction and of the horizontal energy flux.

For the control volume of figure 2, integration is performed for the PME and PTE of (33) and (34). Upon taking the time derivative of these integrated quantities, the temporal rate of change of the total perturbed energy, i.e. $\partial E^*/\partial t^*$ of (24), is obtained. Also, the right-hand side of (24) is rearranged to express $\partial E^*/\partial t^*$ in two parts:

$$\frac{\partial E^*}{\partial t^*} \cong \frac{\partial E_1^*}{\partial t^*} + \frac{\partial E_2^*}{\partial t}, \tag{35}$$

in which

$$E_1^* = \varepsilon \int_0^{z^*} \rho_{00}^* \left(\tfrac{1}{2}\Omega^* r^* v_{rotating}^{*p} + C_p T^{*p}\right) A^* \, \mathrm{d}z^*, \tag{36}$$

$$E_2^* = \varepsilon \int_0^{z^*} \left(\tfrac{1}{2}\rho_{00}^* r^* \Omega^* v_{rotating}^{*p} + h^* \rho^{*p}\right) A^* \, \mathrm{d}z^*. \tag{37}$$

In the above, E_1^* and E_2^* are, respectively, equal to the volume integration of $\frac{1}{2} \times \mathrm{I} + \mathrm{III}$ and $\frac{1}{2} \times \mathrm{I} + \mathrm{II} + \mathrm{IV}$.

Upon substituting (27) into (26) and integrating, one obtains, in view of (14) and (15),

$$\frac{\partial E_1^*}{\partial t^*} = \tfrac{1}{2}\delta W_f^* + \delta Q^*. \tag{38}$$

Also, from (14) and (23),

$$\frac{\partial E_2^*}{\partial t^*} = \tfrac{1}{2}\delta W_f^* - \delta W_p^*. \tag{39}$$

In the light of the earlier considerations of (28), (31) and (32), equation (38) for $\partial E_1^*/\partial t^*$ is concerned with the diffusion in the z^*-direction in the horizontal boundary layer. Equation (39) for $\partial E_2^*/\partial t^*$, on the other hand, expresses the horizontal energy flux. (Note that (39) can also be derived directly from (14) and (15).) The present developments on the behaviour of a rapidly rotating compressible fluid, in response to the given perturbation at the disk, can be summarized as follows (i) Part ($\frac{1}{2}\delta W_f^* + \delta Q^*$) of the externally supplied energy flux is transported in a one-dimensional-like diffusion process in the z^*-direction to the far field across the horizontal boundary layer. In the steady-state limit, the energy flux content e^* is axially uniform at an arbitrary radial location $r^* = r^*$. (ii) The remaining part, ($\frac{1}{2}\delta W_f^*$), of the supplied energy flux is used to induce the horizontal energy flux (δW_p^*), which causes the horizontal flows, and to contribute to the storage of energy ($\partial E_2^*/\partial t^*$) in the boundary layer.

It is a straightforward matter to show that, in the steady-state limit, from (38) and (39),

$$\tfrac{1}{2}\delta W_f^* = -\delta Q^*, \tag{40}$$

$$\tfrac{1}{2}\delta W_f^* = \delta W_p^*. \tag{41}$$

The above equations indicate that half of the mechanical energy flux, $\frac{1}{2}\delta W_f^*$, delivered from the disk to the fluid is recovered at the disk in the form of the rate of thermal energy (δQ^*). The remaining mechanical energy flux from the disk ($\frac{1}{2}\delta W_f^*$) is used to maintain the steady-state horizontal boundary layer flow.

To elaborate the physical implications discussed above, a specific example is considered. Suppose that, at $t=0$, a mechanical perturbation (V_W) and a thermal perturbation (T_W) of time-invariant form are imposed on the disk. The energy flux content e at the disk is

$$e(t, r, \eta = 0) = T_W + 2\alpha^2 r V_W, \tag{42}$$

and, at the initial state $t = 0$, the energy flux content e in the interior fluid is

$$e(t = 0, r, \eta > 0) = 0. \tag{43}$$

Therefore, the solution for e is obtained from the governing equation (31), subject to the boundary and initial conditions, i.e. (42) and (43):

$$e(t, r, \eta) = (T_W + 2\alpha^2 r V_W)\mathrm{erfc}\left(\frac{\eta}{2\sqrt{t}}\right). \tag{44}$$

It follows from (44) that, in the large-time limit, $e(t \to \infty, r, \eta) = T_W + 2\alpha^2 r V_W$ in the entire flow field of the Ekman layer. Note that, in the region far away from the disk ($\eta \gg 1$), the viscous effect is small, and the flow approaches the thermal wind relation, $v = \frac{1}{2} r T$ (see e.g. Sakurai & Matsuda 1974; Matsuda & Hashimoto 1976; Harada 1979). Consequently, at the edge of Ekman boundary layer, the temperature (T) and the azimuthal velocity (v) tend to, respectively,

$$T(t \to \infty, r, \eta \gg 1) = \frac{1}{1 + \alpha^2 r^2}(T_W + 2\alpha^2 r V_W), \tag{45}$$

$$v(t \to \infty, r, \eta \gg 1) = \frac{r}{2(1 + \alpha^2 r^2)}(T_W + 2\alpha^2 r V_W). \tag{46}$$

It is a useful exercise to re-work the above example for the case of an incompressible Boussinesq fluid. The governing equation in this case is a simple heat conduction equation. Furthermore, in the inviscid interior region, the thermal wind relation $v = \frac{1}{2} r T$ prevails (see e.g. Barcilon & Pedlosky 1967*a, b*; Homsy & Hudson 1971). Therefore,

$$T(t, r, \eta) = T_W \mathrm{erfc}\left(\frac{\eta}{2\sqrt{t}}\right), \tag{47}$$

$$T(t \to \infty, r, \eta \gg 1) = T_W, \tag{48}$$

$$v(t \to \infty, r, \eta \gg 1) = \frac{r}{2} T_W. \tag{49}$$

It is evident that, by comparing (44)–(46) with (47)–(49), the energy flux content e for a compressible-fluid flow plays a role analogous to the temperature T for an incompressible Boussinesq-fluid flow. As emphasized previously, the energy transfer from the disk to the interior fluid across the horizontal boundary layer is governed by thermal diffusion for an incompressible Boussinesq fluid. For a compressible fluid, the situation is more complex: the propagation of the perturbation takes a form of a diffusion process of the energy flux content e, which is a combination of the temperature and velocity fields.

As can be seen from (45)–(46), it is important to recognize that, in the steady state, the compressible-fluid flow at the boundary-layer edge ($\eta \gg 1$) is not in isothermal rigid-body rotation. The implication is that, since the far field of the disk ($z \to \infty$) is in isothermal rigid-body rotation, there exists an outer interior flow region of $z \sim O(1)$. Matching between the Ekman layer flow at $z \sim O(E^{1/2})$ and the far-field solution ($z \to \infty$) is effected in this region. The flow in this matching zone is typified by the thermal wind relation $v = \frac{1}{2} r T$. The governing equations and the particular solution for v and T in the outer interior flow region were given previously by Sakurai & Matsuda (1974) and Bark & Bark (1979).

The physical meaning of the outer interior flow region is discussed below. An infinite fluid is considered, which is in rigid-body rotation (at Ω^*) with an infinite disk. In the case of an incompressible homogeneous fluid, when the rotation rate of the disk is suddenly increased from Ω^* to $\Omega^* + \Delta\Omega^*$, the flow near the disk can be described by the Ekman layer solution when $\varepsilon = \Delta\Omega^*/\Omega^* \ll 1$ (see Greenspan 1968). In the far field of the disk, the steady flow is in rigid rotation at Ω^*. Near the disk, the Ekman layer of thickness $O(E^{1/2})$ is present, in which the outer flow rotating at Ω^* and the disk rotation at $\Omega^* + \Delta\Omega^*$ are matched. It is noted that the disturbance in angular momentum, which is caused by the change in rotation rate of the disk ($\Omega^* \to \Omega^* + \Delta\Omega^*$), is transferred to the z-direction by momentum diffusion. At the same time, axial velocity toward the disk is induced by the Ekman pumping, which offsets the momentum diffusion in the z-direction at a distance $O(E^{1/2})$ from the disk. In summary, for an incompressible homogeneous fluid, with only an Ekman layer of thickness $O(E^{1/2})$, the interior fluid outside the Ekman layer can maintain the original rigid-body rotation at Ω^*.

A different picture emerges in the case of a compressible fluid. When the rotation rate of the disk is abruptly altered ($\Omega^* \to \Omega^* + \Delta\Omega^*$), the increased angular momentum near the disk causes a radially outward flow u. Against the background of the basic pressure, which increases exponentially in the radially outward direction, as shown in (2), this radial motion induces a temperature increase by compression work. In short, the external disturbance imposed at the disk gives rise to both momentum diffusion and thermal diffusion in the z-direction. Also, the temperature field shows a dependence on the radial distance. Summarizing these observations for a compressible fluid, by having only the Ekman layer, it is not possible to effecte the matching between the far-field (isothermal rigid-body rotation, $v = T = 0$) and the externally imposed disk wall condition. This can be easily understood by simply noting that the temperature field has a strong r-dependence. Consequently, a portion of the total energy imparted to the disk is transported to the outer interior flow region, which is characterized by $z \sim O(1)$ and $\partial\phi/\partial z \cong \partial\phi/\partial r$, where ϕ denotes a flow variable (see Sakurai & Matsuda 1974; Bark & Bark 1979).

The results of the present analysis are consistent with prior studies. For the case of thermally driven flow, i.e. $T_W = 1$, $V_W = 0$, (45)–(46) recover the results of (3.27) of Matsuda & Sakurai (1974). For the case of mechanically driven flow, i.e. $V_W = r$, $T_W = 0$, (45)–(46) lead to the same result as in (58) and (60) of Harada (1979) (as $\tau \to \infty$, $\eta \gg 1$).

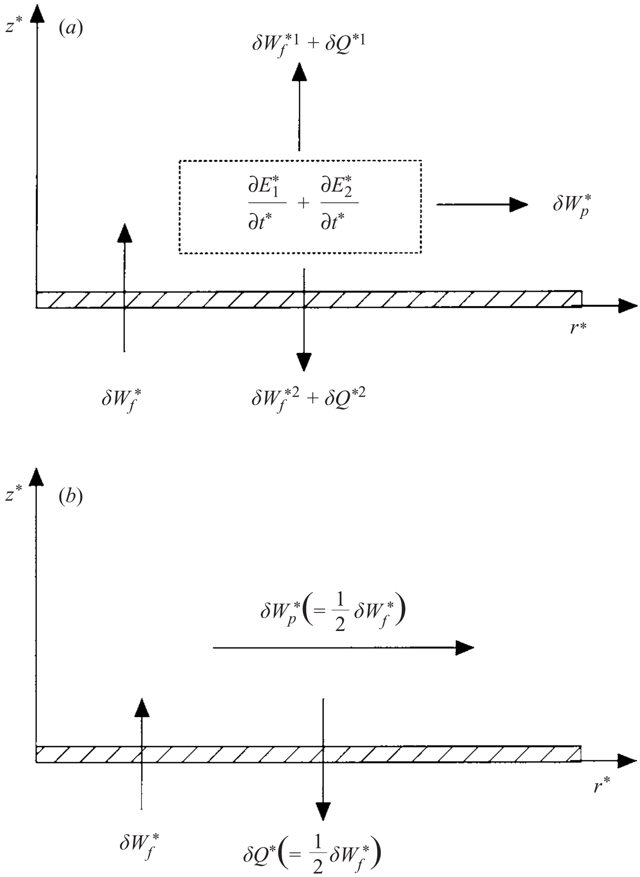

FIGURE 3. Schematic of energy transport in the Ekman layer: (*a*) transient state and (*b*) steady state.

The principal result of the present analysis is that, for a compressible fluid, the momentum diffusion and thermal diffusion across the Ekman layer in the z-direction do not arise independently. The transport process can be effectively described by using the energy flux content, $e[\equiv e^*/(kT_{00}^*)] \equiv T + 2\alpha^2 rv$.

The analysis of energy transport in the present study provides further physical insight. For simplicity, only the case of a mechanical (thermal) perturbation $V_W = r$, $T_W = 0$ ($V_W = 0$, $T_W = 1$) is considered. From (38) and (39), it is seen that the evolutionary process to the large-time steady state has several stages: (i) At $t = 0$, the mechanical (thermal) perturbation $V_W(T_W)$ is imposed at the disk. The momentum (thermal) diffusion in the axial direction from the disk commences. (ii) Due to the axial motion of (i), flow in the radial direction $u > 0$ ($u < 0$) is induced. (iii) The radial motions cause compression (expansion) because of the prevailing pressure gradient (2) of the basic-state. (iv) Thus, a rise (drop) in temperature $T > 0$ ($T < 0$) takes place. (v) The resultant momentum diffusion and thermal diffusion, as depicted by the diffusion of the energy flux content e of (31), bring the fluid to the large-time steady state. These scenarios are illustrated in figure 3(*a*).

It is important to recognize that, in the transient process, part of the mechanical energy externally supplied to the disk is recovered at the disk in the form of thermal energy, i.e. the increase in the temperature of the fluid. Also, as the steady state is approached, the diffusion into the interior vanishes. As shown in (40) and (41), half of the mechanical energy supplied to the disk is reclaimed by the disk in the form of thermal energy. The remaining half is used to maintain the horizontal boundary-layer flow against the prevailing pressure gradient (see figure 3*b*).

5. Conclusions

A transient energy balance analysis has been performed for the transient compressible Ekman layer. Flow is induced by imposing a mechanical and/or thermal disturbance on a rapidly rotating disk in the initial basic state of isothermal rigid-body rotation. The disturbance is a function of time and radial coordinate. The Ekman and Rossby numbers are assumed to be very small.

There are two types of transient energy transfer from the rotating disk to the fluid: (i) The first is a process similar to an incompressible Ekman layer, in which a half of mechanical energy flux ($\frac{1}{2}\delta W_f^*$) is transferred from the rotating disk to the fluid in the development of Ekman layer flow. However, since temperature variations are induced by the radial flow, the Ekman layer of a compressible fluid is more complex than of an incompressible fluid. (ii) The other is a diffusion process. Part of the total energy flux, i.e. ($\frac{1}{2}\delta W_f^* + \delta Q^*$), is transported across the Ekman layer into the interior inviscid region. In this case, the energy transfer is governed by the one-dimensional diffusion equation with the variable $e^* = \frac{1}{2}\Omega^* r^* \mu^* V_W^* + k^* T_W^*$ (in dimensional form), or $e = T + 2\alpha^2 r v$ (in non-dimensional form), which is called the energy flux content. Consequently, the energy flux content turns out to be a physically meaningful variable which plays a role analogous to the temperature in a Boussinesq fluid.

In the steady state, the diffusion of energy flux content into the interior vanishes. Half of the mechanical energy flux ($\frac{1}{2}\delta W_f^*$) supplied to the disk is reclaimed by the disk in the form of thermal energy flux (δQ^*) i.e. $\delta Q^* = -\frac{1}{2}\delta W_f^*$. The remaining half of the mechanical energy flux ($\frac{1}{2}\delta W_f^*$) is used to maintain the horizontal boundary-layer flow against the prevailing pressure gradient, i.e. $\frac{1}{2}\delta W_f^* = \delta W_p^*$.

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